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Microscopic Dynamical Exponents for Random–Random Directed Walk on a One-Dimensional Lattice with Quenched Disorder

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We demonstrate that the dynamical exponent for the time dependence of the coordinate, previously found for an average over disorder, is already present in any realization of a given sample. This ergodicity comes from the existence of a scaling law for the probability distribution of the parameter defining the asymptotic dynamical regime. The self-averaging or non-self-averaging properties of the normal or anomalous phases are direct consequences of this result.

KEY WORDS: Fluctuation phenomena; random processes; Brownian motion; localization in disordered structures.

1. POSITION OF THE PROBLEM AND BASIC EQUATIONS

In a previous paper,[1] we considered the one-dimensional random directed walk on a lattice with *quenched* disorder described by the following master equation:

$$\frac{dp_n}{dt} = -W_n p_n + W_{n-1} p_{n-1} \tag{1}$$

where $p_n(t)$ denotes the probability to be at the site labeled by n at time t. The W's are nonnegative quantities chosen independently at random in

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a given probability distribution $\rho(W)$. We here intend to obtain the dynamical regime at large times for the thermal expectation value of the coordinate, defined as

$$\overline{x(t)} = \sum_{n=0}^{+\infty} n p_n(t) \tag{2}$$

As well known (see, e.g., ref. 2), the dynamics at large times is critically dependent on the relative importance of the bonds having a small W. In order to study quantitatively this effect, we choose $\rho(W)$ as given by

$$\rho(W) = \frac{\mu}{W_m^\mu} W^{\mu-1} \theta(W_m - W) \qquad (W > 0, \quad \mu > 0) \tag{3}$$

where θ is the unit step function. According to the value of μ as compared to 2, one obtains either a standard regime with drift and diffusion (for $\mu > 2$) or a nonstandard one (for $\mu < 2$). For $1 < \mu < 2$, a drift is still present, while the mean square dispersion is superdiffusive. On the other hand, for $\mu < 1$, the motion is wholly anomalous and is characterized by dynamical exponents for the coordinate and the mean-square displacement. All these exponents have been given in ref. 1 for quantities averaged over disorder; there we also explicitly demonstrated that, for $\mu < 1$, $\overline{x(t)}$ is *not* a self-averaging quantity.

The aim of the present paper is to show that, at least for the coordinate, the same exponent is present at a "microscopic" level, i.e., does arise in a *given sample*. In ref. 1, we stated that the average over disorder qualitatively changes the behavior at large times of the probability $p_0(t)$ to be at time t at the starting point. This phenomenon does not hold for $\overline{x(t)}$. Indeed, we establish below that, for any μ, one has for a given sample

$$\overline{x(t)} \sim x_0 (W_m t)^\alpha \tag{4}$$

where α is a nonrandom exponent, whereas x_0 is a random variable following a probability distribution law $p_\mu(x_0)$. In the following, we find $p_\mu(x_0)$ which is fully specified by the knowledge of all its positive moments [see Eq. (22) below]. The non-self-averaging property of $\overline{x(t)}$ for $\mu < 1$ originates from the fluctuations of the random number x_0. On the contrary, for $\mu > 1$, we shall show that x_0 takes a single value with probability 1, which is consistent with the fact that $\overline{x(t)}$ is then self-averaging as time goes on. In addition, one has $\alpha = 1$ in this case.

The problem is conveniently solved by the use of Laplace transforms. We set

$$x_1(z) = \int_0^{+\infty} e^{-zt} \overline{x(t)} \, dt, \qquad \Gamma(z) = z^2 x_1(z) \tag{5}$$

where $\Gamma(z)$ is thus the Laplace transform of the acceleration $d^2\overline{x(t)}/dt^2$ and is a functional of all the W's realized in a given sample. It is easily seen that the following functional equation holds:

$$\Gamma(z, W_0, W_1, W_2,...) = \frac{W_0}{z + W_0} [z + \Gamma(z, W_1, W_2, W_3,...)] \qquad (6)$$

Note that Γ is a positive quantity for z a real positive number. The same kind of relation was used in ref. 1 to get a closed explicit expression for quadratic moments [see Eqs. (14) and (15) in that paper]. From this relation one deduces that the probability distribution for the random variable Γ, $P(\Gamma, z)$, obeys the following integral equation:

$$P(\Gamma, z) = \int_0^{+\infty} dW \rho(W) \frac{z+W}{W} P\left(\frac{z+W}{W}\Gamma - z, z\right) \qquad (7)$$

Since Γ is a positive quantity, $P(\Gamma, z)$ identically vanishes for $\Gamma < 0$. Once $P(\Gamma, z)$ is known, the probability density function for $x_1(z)$, $Q(x_1, z)$, can be obtained by the use of the relation

$$Q(x_1, z) = z^2 P(z^2 x_1, z) \qquad (8)$$

Before entering into the details of the calculation, a comment is in order. As can be seen by iteration of Eq. (6), Γ is a so-called Kesten's variable.[3] However, since $|W_n/(z + W_n)| < 1$, the Kesten equation

$$\langle [W/(z+W)]^\kappa \rangle = 1$$

has the unique trivial solution $\kappa = 0$. In other words, Kesten's theorem does not apply here.

2. CALCULATION OF $Q(x_1, z)$ AND CONSEQUENCES

In order to analyze the integral equation (7), we first Laplace transform it with respect to Γ by defining

$$\Pi(\tau, z) = \int_0^{+\infty} d\Gamma\, e^{-\Gamma\tau} P(\Gamma, z)$$

Direct substitution in Eq. (7) yields

$$\Pi(\tau, z) = \int_0^{+\infty} dW \rho(W)\, e^{-Wz\tau/(z+W)} \Pi\left(\frac{W}{z+W}\tau, z\right) \qquad (9)$$

This latter integral equation is now formally solved by assuming an entire series expansion for $\Pi(\tau, z)$:

$$\Pi(\tau, z) = 1 + \sum_{n=1}^{+\infty} \alpha_n(z)\,\tau^n \tag{10}$$

Note that $\alpha_n(z) = (-1)^n n! \langle \Gamma^n \rangle_P(z)$. Thus, we are assuming that all the positive moments of $P(\Gamma, z)$ exist, a fact which will be established below. From Eq. (9), the coefficients are seen to obey the following triangular recursion:

$$\alpha_n(z) = \frac{S_n(z)}{1 - S_n(z)} \sum_{m=1}^{n} \frac{(-1)^m}{m!} z^m \alpha_{n-m}(z), \qquad \alpha_0(z) = 1 \tag{11}$$

where the quantities $S_n(z)$ are defined as

$$S_n(z) = \langle [W/(z+W)]^n \rangle$$

By using Eq. (3), it is seen that $S_n(z)$ has the expansion

$$S_n(z) = 1 - \frac{\pi}{\sin \pi\mu} \frac{(z/W_m)^\mu}{B(n, \mu)} - \mu \sum_{p=1}^{+\infty} \frac{n(n+1)\cdots(n+p-1)}{p!(p-\mu)} \left[\frac{-z}{W_m}\right]^p + \cdots \tag{12}$$

where $B(n, \mu)$ denotes as usual the beta function $\Gamma(n)\,\Gamma(\mu)/\Gamma(n+\mu)$. It appears hopeless to solve in a closed form the recursion (11) for any z. However, we can find the asymptotic form of $\alpha_n(z)$ for $|z| \ll W_m$; indeed, according to Eqs. (11) and (12), it is seen that the $\alpha_n(z)$ have the following approximate expression:

$$\alpha_n(z) = \frac{(-1)^n z^n}{(1-S_1)(1-S_2)\cdots(1-S_n)} \left\{1 + O\left[\left(\frac{z}{W_m}\right)^\beta\right]\right\} \tag{13}$$

where β is a positive exponent.

This obviously yields the small-z behavior of $\Pi(\Gamma, z)$ and thus allows one to obtain the probability distribution of $\overline{x(t)}$ at large times. For clarity, we now investigate separately the two cases $\mu < 1$ and $\mu > 1$. Their differences result from the behavior of $\alpha_1(z)$ at small z; indeed, from (3) one has

$$\text{(i)}\quad \mu < 1: \qquad \langle (z+W)^{-1} \rangle = W_m^{-\mu} \frac{\pi\mu}{\sin \pi\mu} z^{\mu-1} + \cdots \tag{14}$$

$$\text{(ii)}\quad \mu > 1: \qquad \langle (z+W)^{-1} \rangle = W_m^{-1} \frac{\mu-1}{\mu} + \cdots \equiv \left\langle \frac{1}{W} \right\rangle + \cdots \tag{15}$$

2.1. $\mu < 1$

By using Eqs. (12)–(14), it then turns out that, for $|z| \ll W_m$, the α_n are given to the leading order by

$$\alpha_n(z) \approx (-1)^n \left[\left\langle \frac{1}{z+W} \right\rangle\right]^{-n} \frac{1!\,2!\,3!\cdots(n-1)!}{(\mu+1)^{n-1}(\mu+2)^{n-2}\cdots(\mu+n-1)} \quad (n>1)$$

$$\alpha_1(z) \approx -\left[\left\langle \frac{1}{z+W} \right\rangle\right]^{-1} \tag{16}$$

From the above equations, it is readily seen that for small z, $\Pi(\tau, z)$ can be written as

$$\Pi(\tau, z) = F_\mu[Z(\tau, \mu)], \qquad Z(\tau, \mu) = \frac{\sin \pi\mu}{\pi\mu} W_m^\mu z^{1-\mu}\tau \tag{17}$$

where the function $F_\mu(Z)$ is given by the expansion

$$\begin{aligned} F_\mu(Z) &= 1 - Z + \sum_{n=2}^{+\infty} \frac{1!\,2!\cdots(n-1)!}{(\mu+1)^{n-1}(\mu+2)^{n-2}\cdots(\mu+n-1)} (-Z)^n \\ &\equiv \sum_{n=0}^{+\infty} c_n(-Z)^n \end{aligned} \tag{18}$$

Equations (17) and (18) show that $\Pi(\tau, z)$ is a series of the form $\sum_n d_n(z^{1-\mu})^n$. In a full calculation, d_n should be replaced by some function of z, the leading term of which is precisely d_n, whereas the first correction is of the order $(z/W_m)^\mu$ or $(z/W_m)^{1-\mu}$ (see Section 3). For the asymptotic regime, the approximation given in (18) is sufficient.

The scaling law provided by (17) and (18) establishes the fact that, for any sample, $\overline{x(t)}$ behaves like t^μ. Indeed, let a random function $\phi(t)$ be such that $\phi(t) = at^\mu$, where a is random and follows the probability distribution $\omega(a)$. Then, the Laplace transform of $\phi(t)$ is $\Phi(z) = a\Gamma(\mu+1)\, z^{-(\mu+1)}$; the probability distribution of Φ is simply given by

$$\Omega(\Phi, z) = \frac{Z^{\mu+1}}{\Gamma(\mu+1)} \omega\left[\frac{Z^{\mu+1}}{\Gamma(\mu+1)} \Phi\right] \tag{19}$$

Denoting now by $f_\mu(X)$ the Laplace inverse of the function $F_\mu(Z)$ defined in Eq. (18), using the above scaling law (17) and Eq. (8), we find

$$\overline{x(t)} \sim x_0(W_m t)^\mu \tag{20}$$

where x_0 is a random number with a distribution $p_\mu(x_0)$ given by

$$p_\mu(x_0) = \frac{\pi\mu}{\sin \pi\mu} \Gamma(\mu+1) f_\mu \left[\frac{\pi\mu}{\sin \pi\mu} \Gamma(\mu+1) x_0 \right] \tag{21}$$

$p_\mu(x_0)$ is explicitly known by all its moments; indeed, one has

$$\langle x_0^n \rangle = \int_0^{+\infty} x_0^n p_\mu(x_0)\, dx_0 = \langle x_0 \rangle^n \frac{1!\, 2! \cdots n!}{(\mu+1)^{n-1} (\mu+2)^{n-2} \cdots (\mu+n-1)} \tag{22}$$

where

$$\langle x_0 \rangle = \frac{\sin \pi\mu}{\pi\mu\Gamma(\mu+1)} \tag{23}$$

Note that Eqs. (20) and (23) reproduce the disorder average given by Eq. (21) in ref. 1, as they should. All the moments can be easily calculated in a recursive way due to the obvious relation

$$\langle x_0^{n+1} \rangle = \frac{(n+1)!}{(\mu+1)(\mu+2)\cdots(\mu+n)} \langle x_0 \rangle \langle x_0^n \rangle \tag{24}$$

It is interesting to observe that $p_\mu(x_0)$ is *not* a broad law in the sense that all its positive moments exist. Indeed, due to the fact that the series given by Eq. (18) is a convergent one for any finite Z and $\mu > 0$, it can be inferred that $p_\mu(x_0)$ certainly decreases faster than a stretched exponential $\exp(-x_0^\alpha)$ $(\alpha > 0)$ at large x_0. This is consistent with the fact that Kesten's theorem is not applicable in our case. Note that $p_\mu(x_0)$ takes on a very simple form for $\mu = 0$ or $\mu = 1$:

(i) $\mu = 0$ $\quad F_0(Z) = (1+Z)^{-1}$ $\quad$ i.e., $p_{\mu=0}(x_0) = e^{-x_0}$

(ii) $\mu = 1$ $\quad F_1(Z) = e^{-Z}$ $\quad$ i.e., $p_{\mu=1}(x_0) = \delta(x_0 - 0^+)$

Note that the value $\mu = 0$ is not strictly allowed, since in this case, the repartition law $\rho(W)$ [see Eq. (3)] would not be normalized. There is a qualitative discontinuous change between $\mu = 0$ and $\mu = 0^+$, which is reflected by the fact that for any $\mu > 0$, $F_\mu(Z)$ has no singularity at a finite distance of the origin, whereas for $\mu = 0$, a unique pole arises for $Z = -1$. In this respect, the purely exponential function e^{-x_0} cannot properly represent the limiting situation $\mu = 0^+$, for which the value of $p(x_0 = 0^+)$ is conjectured to be equal to 0.5, and not to 1.

The second result above means that, for $\mu = 1$, the random number x_0 takes the value 0^+ with probability 1. This is in agreement with the fact that, in this case, the velocity indeed vanishes and that $\mu = 1$ is the onset of the self-averaging property for the velocity [see Eqs. (22) and (30) in ref. 1]. For $0 < \mu < 1$, $F_\mu(Z)$ smoothly interpolates between $F_0(Z)$ and $F_1(Z)$ (see Fig. 1).

A better insight into the distribution $p_\mu(x_0)$ is provided by the two first moments. From Eq. (23), one sees that the average value of x_0 decreases from 1 to zero when μ increases from 0 to 1 (see Fig. 1). It is readily seen that

$$\mu \to 0^+ \qquad \langle x_0 \rangle \to 1 + C\mu \qquad (C = \text{Euler's constant})$$

$$\mu \to 1 \qquad \langle x_0 \rangle \to 1 - \mu$$

The mean square deviation is given by

$$\langle x_0^2 \rangle - \langle x_0 \rangle^2 = \langle x_0 \rangle^2 \frac{1-\mu}{1+\mu} \tag{25}$$

and displays the same monotonic variation as $\langle x_0 \rangle$ (see Fig. 2):

$$\mu \to 0^+ \qquad \langle x_0 \rangle \to 1 - 2(C+1)\,\mu$$

$$\mu \to 1 \qquad \langle x_0 \rangle \to (1-\mu)^3$$

On the contrary, higher cumulants do not have such a plain variation and display oscillationlike behavior.

The function $p_\mu(x_0)$ has been numerically computed according to the following scheme. Due to the fact that $F_\mu(Z)$ has no singularity, for $\mu > 0$,

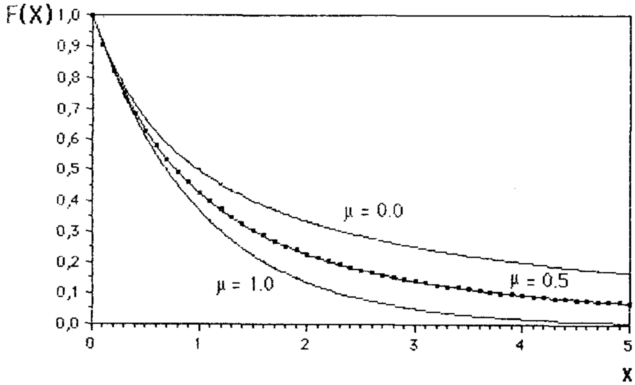

Fig. 1. Variation of the scaling function $F_\mu(X)$ as defined by Eq. (18) for $\mu = 0.0$, 0.5, 1.0.

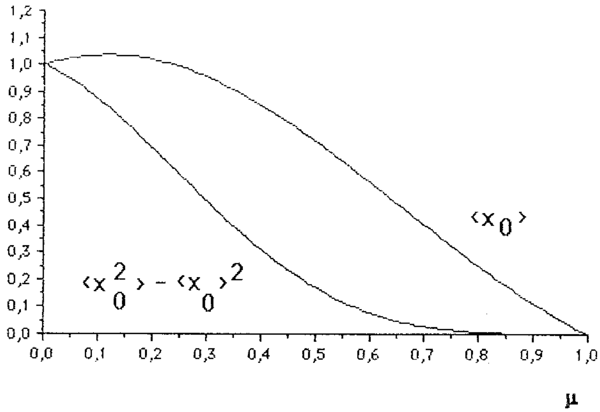

Fig. 2. Variation of the first moment and of the mean square dispersion of the probability distribution $p_\mu(x_0)$ giving the random coefficient x_0 of $\overline{x(t)}$ in the asymptotic regime for $\mu < 1$ [see Eqs. (22), (23), and (25)].

in the closed half-plane Re $Z \geqslant 0$, the Bromwich line can be shifted onto the imaginary axis. The inverse Laplace transformation thus takes the form

$$p_\mu(x_0) = \frac{1}{\langle x_0 \rangle} \int_0^{+\infty} dt \left[F_+(t) \cos \frac{x_0}{\langle x_0 \rangle} t + F_-(t) \sin \frac{x_0}{\langle x_0 \rangle} t \right]$$

where $F_\pm$ denotes the even and odd parts of $F_\mu(Z = it)$:

$$F_+(t) = \tfrac{1}{2}[F_\mu(it) + F_\mu(-it)] = \sum_{p=0}^{+\infty} (-1)^p c_{2p} t^{2p}$$

$$F_-(t) = \tfrac{1}{2}[F_\mu(it) - F_\mu(-it)] = \sum_{p=0}^{+\infty} (-1)^p c_{2p+1} t^{2p+1}$$

The coefficients c_n are given by the expansion (18). $p_\mu(x_0)$ can then be numerically computed by first summing the series and then performing a numerical quadrature. The results are reported on Fig. 3 for several values of μ. It is seen that, as expected, when μ increases, a peak occurs which is more and more pronounced and moves toward the origin. In the limit $\mu \to 1$, this fact yields the $\delta(x_0 - 0^+)$ distribution. This phenomenon may be viewed as the precursor of the settling of the self-averaging property at $\mu = 1$ (see Fig. 3 in ref. 1).

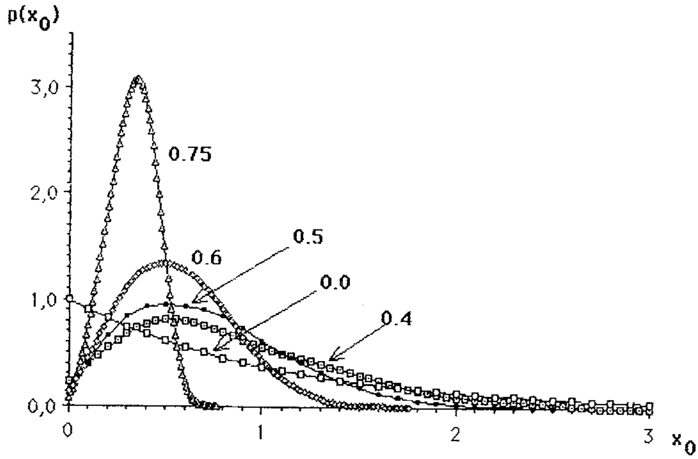

Fig. 3. Variation of the distribution function p_μ as a function of x_0 for $\mu = 0.0$, 0.4, 0.5, 0.6, and 0.75. Recall that, *stricto sensu*, the curve $\mu = 0$ does not belong to the class of models considered here.

2.2. $\mu > 1$

Now, due to Eqs. (12), (13), and (15), one finds

$$\alpha_n(z) = \frac{(-1)^n}{n!} \langle W^{-1} \rangle^{-n} \tag{26}$$

which in turn implies that

$$\Pi(\tau, z) = \exp(-\tau/\langle W^{-1} \rangle) \tag{27}$$

Thus, for small z, $P(\Gamma, z)$ is given by

$$P(\Gamma, z) = \delta(\Gamma - 1/\langle W^{-1} \rangle) \tag{28}$$

This shows that the limit of the derivative of $\overline{x(t)}$ with respect to time, i.e., the velocity, tends to $\langle W^{-1} \rangle^{-1}$ with probability 1 at large times. This thus quickly establishes the existence of a finite ordinary drift characterized by a self-averaging velocity, a result already obtained in ref. 4 by other methods.

3. RELEVANT TIME SCALE

On physical grounds, it is important to find the time scale t_1 beyond which the asymptotic regime characterized by Eq. (20) is indeed displayed in a given sample. It can be guessed that, in the anomalous phase $\mu < 1$, this time should exhibit a minimum. Indeed, in order to achieve this asymptotic regime, the particle has to experience properly the surrounding disorder, i.e., to feel the existence of many nearly broken links. For $\mu \to 0^+$, these links are relatively numerous, but, since when a single small W is met, it takes a very large time (of the order of $1/W$) to go ahead, it will take a long time to see many such links. Thus, a very long time has to elapse before the anomalous asymptotic regime can occur. This can be viewed as a precursor of the ultraslow Sinai diffusion which occurs at $\mu = 0$ in the general walk. On the other hand, for $\mu \to 1$, the particle moves quasinormally ($\sim t$) before encountering quasibroken links in sufficient number. It thus again takes a very long time to experience them, since those links are not very numerous.

One way to find an estimate of the time scale t_1 is to analyze the first correction to the probability distribution $\Pi(\tau, z)$, the dominant term of

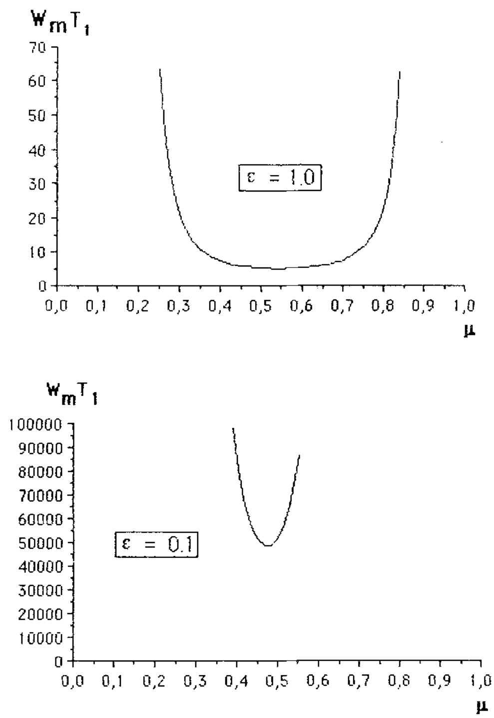

Fig. 4. Variation of the time scale $W_m t_1$ as defined in Section 3; each curve is labeled by the value of ε.

which is given by the scaling function $F_\mu(Z)$ as defined by Eq. (18). After a tedious algebra, it is found that $\Pi(\tau, z)$ can be written as

$$\Pi(\tau, z) = F_\mu(Z) - \frac{\pi}{2 \sin \pi\mu} \left(\frac{z}{W_m}\right)^\mu G_\mu(Z) + \frac{\mu}{1-\mu} \frac{\sin \pi\mu}{\pi} \left(\frac{z}{W_m}\right)^{1-\mu} H_\mu(Z) \tag{29}$$

where G_μ and H_μ are known convergent series. In order to provide an estimate for t_1, we take G_μ and H_μ of the order of unity and we define the first correction Δ as

$$\Delta = \frac{\pi}{2 \sin \pi\mu} \left(\frac{z}{W_m}\right)^\mu + \frac{\mu}{1-\mu} \frac{\sin \pi\mu}{\pi} \left(\frac{z}{W_m}\right)^{1-\mu} \tag{30}$$

Since $F_\mu(z)$ is also assumed to be of the order of unity, one requires that Δ be a small number ε. By writing $\Delta = \varepsilon$, we find $Z_1(\mu, \varepsilon)$ and eventually $t_1(\mu, \varepsilon) = 1/Z_1(\mu, \varepsilon)$. Thus, for times $t \gg t_1$, the regime described by Eq. (20) should be observable. Figure 4 shows the variation of $t_1(\mu, \varepsilon)$ as a function of μ for $\varepsilon = 1$ and $\varepsilon = 0.1$. Clearly, the time to enter the asymptotic regime, if properly described by the above t_1, is indeed very large, as expected.

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